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ON RATIONAL INTEGRABILITY OF EULER EQUATIONS ON LIE ALGEBRA $\mathrm{so}(4,\mathbb{C})$

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ABSTRACT

We consider the Euler equations on the Lie algebra $\mathrm{so}(4,\mathbb{C})$ with a diagonal quadratic Hamiltonian. It is known that this system always admits three functionally independent polynomial first integrals. We prove that if the system has a rational first integral functionally independent of the known three ones so called fourth integral, then it has a polynomial first integral that is also functionally independent of them. This is a consequence of more general fact that for these systems the existence of Darboux polynomial with no vanishing cofactor implies the existence of polynomial fourth integral.

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1. Introduction

For a given system of (polynomial) ordinary differential equations depending on parameters, the question arises, how to recognize those values of the parameters for which the equations have (rational or polynomial) first integrals? Except for some simple cases, this problem is very hard and there are no satisfying methods to solve it.

In this paper we obtain a partial result concerning this problem relevant for the so-called **Euler equations on Lie algebras** [1–3, 6, 13, 14]. For these equations the problem is largely open too.

Let us recall their definition. Let $(L, [\cdot,\cdot])$ be a finite dimensional (real or complex) Lie algebra. L^* its dual. For $f, g \in C^\infty(L^*)$ their **Lie-Poisson bracket** $\{f, g\}$ is defined by

$$\{f, g\}(x) = \langle x, [df(x), dg(x)]\rangle,$$

where $x \in L^*$, $df(x)$, $dg(x) \in (L^*)^* = L^{**} \approx L$, and where for $x \in L^*$ and $y \in L$, $\langle x, y\rangle = x(y)$.

Recall that the function $F \in C^\infty(L^*)$ is a **Casimir function** of the Lie algebra L if $\{f, F\} = 0$ for every $f \in C^\infty(L^*)$.

Element $x \in L^*$ can be written $x = \sum_{i=1}^n x_i e_i^*$; $x_i \in C^\infty(L^*)$, $1 \le i \le n$, where $\{e_1^*, \ldots, e_n^*\}$ is the basis dual to a fixed basis $\{e_1, \ldots, e_n\}$ of L.

For a given function $H \in C^\infty(L^*)$, the system of differential equations

$$\frac{dx_i}{dt} = \{x_i, H\}, \quad 1 \le i \le n, \tag{1.1}$$

is called **Euler equations on the Lie algebra** L with the **Hamiltonian** H.

It is easy to see, [13], that a function F defined on L^* is a first integral of system (1.1) if and only if $\{F, H\} = 0$. In particular, the Hamiltonian H and any Casimir function of the Lie algebra L are first integrals of system (1.1).

Only for Hamiltonians H that are functionally independent of the Casimir functions, the right sides of system (1.1) do not vanish identically. That is why we will always suppose that the Hamiltonian and Casimir functions are functionally independent.

From now on we will concentrate only on complex six dimensional Lie algebra so$(4, \mathbb{C})$—the Lie algebra of the complex Lie group SO$(4,\mathbb{C})$ and study one of the simplest examples of Euler equations on it—the Euler equations corresponding to the so called **diagonal quadratic Hamiltonian**.

The Lie algebra $\mathrm{so}(4,\mathbb{C})$ admits two functionally independent polynomial Casimir functions. Thus any system of Euler equations on it always admits three functionally independent first integrals.

For this Lie algebra, on the level manifolds of two functionally independent Casimir functions any Euler system, at least locally, can be reduced to the standard Hamiltonian equations with two degrees of freedom (see Sections 6.1–6.2 and Theorem 6.22 from [13]).

In appropriate basis of Lie algebra $\mathrm{so}(4,\mathbb{C})$ (see [1]), the Euler equations corresponding to a diagonal quadratic Hamiltonian $\frac{1}{2}\sum_{i=1}^{6}\lambda_i x_i^2$, take the following elegant form:

$$
\begin{aligned}
\frac{dx_1}{dt} &= (\lambda_3-\lambda_2)x_2x_3 + (\lambda_6-\lambda_5)x_5x_6,\\
\frac{dx_2}{dt} &= (\lambda_1-\lambda_3)x_1x_3 + (\lambda_4-\lambda_6)x_4x_6,\\
\frac{dx_3}{dt} &= (\lambda_2-\lambda_1)x_1x_2 + (\lambda_5-\lambda_4)x_4x_5,\\
\frac{dx_4}{dt} &= (\lambda_3-\lambda_5)x_3x_5 + (\lambda_6-\lambda_2)x_2x_6,\\
\frac{dx_5}{dt} &= (\lambda_4-\lambda_3)x_3x_4 + (\lambda_1-\lambda_6)x_1x_6,\\
\frac{dx_6}{dt} &= (\lambda_2-\lambda_4)x_2x_4 + (\lambda_5-\lambda_1)x_1x_5,
\end{aligned}
\tag{1.2}
$$

where $\lambda := (\lambda_1,\dots,\lambda_6)\in\mathbb{C}^6$. Exactly the same construction takes place for Lie algebra $\mathrm{so}(4,\mathbb{R})$, where $\lambda := (\lambda_1,\dots,\lambda_6)\in\mathbb{R}^6$ and equations (1.2) remain unchanged.

They always have three first integrals:

$$
H_1 = x_1x_4 + x_2x_5 + x_3x_6, \quad H_2 = \sum_{i=1}^{6} x_i^2, \quad H_3 = \sum_{i=1}^{6}\lambda_i x_i^2. \tag{1.3}
$$

Unless all the λ_i, $1\le i\le 6$, are equal, in which case the right hand sides of system (1.2) vanish, these three integrals are functionally independent.

The first integrals H_1 and H_2 are Casimir functions of the Lie algebra $\mathrm{so}(4,\mathbb{C})$.

Whatever the chosen notion of integrability, the system (1.2), to be integrable needs a supplementary first integral H_4, functionally independent of H_1, H_2 and H_3, called shortly a **fourth integral**. The only known cases when the fourth integral exists are the **Manakov case**, defined by the condition

$$M = \lambda_1\lambda_4(\lambda_2+\lambda_5-\lambda_3-\lambda_6)+\lambda_2\lambda_5(\lambda_3+\lambda_6-\lambda_1-\lambda_4)+\lambda_3\lambda_6(\lambda_1+\lambda_4-\lambda_2-\lambda_5) = 0,$$

and the **product case**, defined by the conditions

$$\lambda_1 = \lambda_4, \quad \lambda_2 = \lambda_5, \quad \lambda_3 = \lambda_6.$$

In both cases the fourth integral can be found among the polynomials of degree at most 2 (see [1, 9]). As in [9] the table of these first integrals was not correctly printed, for the sake of completeness we reproduce its correct form in Appendix.

We will concentrate only on **fourth rational integrals**. As is well-known, their absence implies the absence of algebraic fourth integrals [8, 18, 19] as well as the absence of meromorphic fourth integrals defined on some neighbourhood of 0 of $\mathbb{C}^6$, [20].

The main aim of this paper is to prove the following theorem.

THEOREM 1.1: *If for some $\lambda \in \mathbb{C}^6$, the Euler equations* (1.2) *admit a rational fourth integral, then they admit a polynomial fourth integral.*

Let us note that from the validity of Theorem 1.1 in complex setting, its validity in real one follows immediately.

The proof of Theorem 1.1 is based on the study of **Darboux polynomials** (see Section 2.1) for Euler equations (1.2) and the rich symmetry properties of these equations. In fact, Theorem 1.1 is a direct consequence of the more general statement (Theorem 3.1) concerning these polynomials.

Let us underline that the following conjecture remains open.

Conjecture : In both cases, $\mathrm{so}(4, \mathbb{C})$ and $\mathrm{so}(4, \mathbb{R})$, Euler equations (1.2) have a polynomial fourth integral only either in the Manakov case or in the product case.

See [1, 3, 5, 6, 15, 17] for partial results which confirm it.

The paper is organized as follows. In Section 2 we collect all auxiliary facts needed for the proof. In Section 3 Theorem 1.1 is obtained as a direct consequence of more general Theorem 3.1 concerning Darboux polynomials. Let us stress that all proofs are completely elementary.

Finally let us note that in [10] an exact counterpart of Theorems 1.1 and 3.1 is proved for so called natural polynomial hamiltonian systems of arbitrary degree of freedom.

2. Preliminaries

2.1. Darboux polynomials. Consider a polynomial system of ordinary differential equations defined in $\mathbb{C}^n$

$$\frac{dx_j}{dt} = V_j(x_1, \dots, x_n), \quad 1 \le j \le n. \tag{2.1}$$

For a holomorphic function F defined on some open subset of $\mathbb{C}^n$ let us define

$$d(F) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} V_i.$$

The operator d is called a **derivation** associated with system of differential equations (2.1).

A polynomial $P \in \mathbb{C}[x_1, \dots, x_n] \setminus \mathbb{C}$ is called a **Darboux polynomial** of system (2.1) if for some polynomial $S \in \mathbb{C}[x_1, \dots, x_n]$ one has

$$d(P) = SP. \tag{2.2}$$

The polynomial S is called a **cofactor** of the Darboux polynomial P. When $S \neq 0$, P is called a **proper** Darboux polynomial. When $S = 0$, P is nothing but a first integral of system (2.1).

Here we mention some properties of the Darboux polynomials:

- (D1) Let P_1 and P_2 be non-zero relatively prime polynomials that are not first integrals of system (2.1). Then the rational function P_1/P_2 is a first integral of system (2.1) if and only if P_1 and P_2 are its proper Darboux polynomials with the same cofactor.
- (D2) All factors of a Darboux polynomial of system (2.1) are also its Darboux polynomials.
- (D3) If P_1 and P_2 are two Darboux polynomials of system (2.1) with cofactors S_1 and S_2, respectively, then $P_1 P_2$ is also its Darboux polynomial with cofactor $S_1 + S_2$.
- (D4) Let us suppose that the right-hand sides of system (2.1) are homogeneous polynomials of the same degree. Let P be a Darboux polynomial

of system (2.1). Then its cofactor S is homogeneous and all homogeneous components of P are also Darboux polynomials of system (2.1).

See [12] for more details.

2.2. Permutational symmetries. The Euler equations (1.2) possess invariant property called **permutational symmetry**. The permutational symmetries can be described generally as follows. Let $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, and let $V(x, \lambda) = (V_1(x, \lambda), \ldots, V_n(x, \lambda))$ depend holomorphically on $(x, \lambda) \in \mathbb{C}^{2n}$. Let us consider the following system of differential equations

$$\frac{dx}{dt} = V(x, \lambda). \tag{2.3}$$

Let σ be an element of the symmetric group S_n, i.e., the group of all permutations of $\{1, \ldots, n\}$. For $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ we will note

$$\sigma(a) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)}).$$

A permutation $\sigma \in \mathrm{S}_n$ will be called a **permutational symmetry** of system (2.3) if for all $(x, \lambda) \in \mathbb{C}^{2n}$, one has

$$V_k(\sigma(x), \sigma(\lambda)) = \varepsilon V_{\sigma(k)}(x, \lambda), \quad 1 \leq k \leq n,$$

where $\varepsilon = \pm 1$ is a constant independent of k. All permutational symmetries of system (2.3) form a group.

Theorem 2.1: *Let σ be a permutational symmetry of system* (2.3).

(a) Let $F = F(x)$ be a first integral of system (2.3). Then the function $\tilde{F} = F \circ \sigma^{-1}$ is a first integral of the system

$$\frac{dx}{dt} = V(x, \sigma(\lambda)). \tag{2.4}$$

(b) Let $P = P(x)$ be a Darboux polynomial of system (2.3) (see (2.2)). Let us note $\tilde{d}$ the derivation associated with system (2.4). Then

$$\tilde{d}(\tilde{P}) = \tilde{S}\tilde{P},$$

where $\tilde{P} = P \circ \sigma^{-1}$ and $\tilde{S} = S \circ \sigma^{-1}$.

For the proof of (a) see Section II of [9]. (b) is proved exactly along the same lines.

The group of permutational symmetries of the Euler equations (1.2) consists of 24 elements. Among others it contains the following five permutations:

$$\begin{aligned} \tau_2(1,2,3,4,5,6) &= (2,1,3,5,4,6),\\ \tau_3(1,2,3,4,5,6) &= (3,2,1,6,5,4),\\ \tau_4(1,2,3,4,5,6) &= (4,2,6,1,5,3),\\ \tau_5(1,2,3,4,5,6) &= (5,4,3,2,1,6),\\ \tau_6(1,2,3,4,5,6) &= (6,2,4,3,5,1). \end{aligned} \tag{2.5}$$

For more details see Section II of [9] where, with the notations therein, $\tau_2 = \sigma_1$, $\tau_3 = \sigma_3$, $\tau_4 = \sigma_7$, $\tau_5 = \sigma_8 \circ \sigma_1$ and $\tau_6 = \sigma_7 \circ \sigma_3$.

Let P be a proper Darboux polynomial of system (1.2), that is $d(P) = SP$, where d is the corresponding derivation and $S \in \mathbb{C}[x_1,\dots,x_6]\backslash\{0\}$, $S(x) = \sum_{i=1}^{6} \alpha_i x_i$, $\alpha_1,\dots,\alpha_6 \in \mathbb{C}$ and at least one of them is non-zero, say $\alpha_{i_0} \neq 0$. According to (2.5) $\tau_{i_0}(i_0) = 1$. Now, Theorem 2.1b implies that without loss of generality, one can always assume that $\alpha_1 \neq 0$. This fact will be used in the proof of Theorem 1.1.

Further, d will always denote the derivation associated with the Euler equations (1.2).

2.3. Another invariance property. Beside permutational symmetries, the Euler equations (1.2) possess also another invariant property related to the change of signs of the couples of variables (x_1,x_4), (x_2,x_5) and (x_3,x_6) respectively. More precisely, let us note that

$$\begin{aligned} \tau_{14}(x_1,x_2,x_3,x_4,x_5,x_6) &= (-x_1,x_2,x_3,-x_4,x_5,x_6),\\ \tau_{25}(x_1,x_2,x_3,x_4,x_5,x_6) &= (x_1,-x_2,x_3,x_4,-x_5,x_6),\\ \tau_{36}(x_1,x_2,x_3,x_4,x_5,x_6) &= (x_1,x_2,-x_3,x_4,x_5,-x_6). \end{aligned} \tag{2.6}$$

It is easy to see that for $(ij) = (14)$, $(ij) = (25)$ and $(ij) = (36)$,

$$\tau_{ij}{}^{-1} \circ d \circ \tau_{ij} = -d,$$

that means that under these transformations, the right hand sides of equations (1.2) change signs.

For the polynomial $T \in \mathbb{C}[x_1,\dots,x_6]$, let us denote $T_{(ij)} := T \circ \tau_{ij}$. Thus, if T is a first integral of the system (1.2), then $T_{(14)}$, $T_{(25)}$ and $T_{(36)}$ are also first integrals of this system.

Moreover, if P is its Darboux polynomial, that is $d(P) = SP$, then

$$d(P_{(ij)}) = -S_{(ij)}P_{(ij)}.$$

In particular, if

$$d(P)(x) = (\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5 + \alpha_6 x_6)P(x), \tag{2.7}$$

then

$$d(P_{(14)})(x) = (\alpha_1 x_1 - \alpha_2 x_2 - \alpha_3 x_3 + \alpha_4 x_4 - \alpha_5 x_5 - \alpha_6 x_6)P_{(14)}(x) \tag{2.8}$$

and

$$d(P_{(25)})(x) = (-\alpha_1 x_1 + \alpha_2 x_2 - \alpha_3 x_3 - \alpha_4 x_4 + \alpha_5 x_5 - \alpha_6 x_6)P_{(25)}(x). \tag{2.9}$$

2.4. Explicit form of some linear differential operators. Let us denote by X_{ij}, $1 \le i < j \le 6$, the linear differential operator defined by the formula

$$X_{ij}(G) = \det \frac{\partial(H_1, H_2, H_3, G)}{\partial(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_6)},$$

where G is a holomorphic function and $\hat{x}_r$ means the absence of x_r.

These operators play a crucial role in the proof of Theorem 1.1. In particular, for this proof we need the explicit formula for some of them.

To simplify notation, we write: $\lambda_{ij} = \lambda_i - \lambda_j$ for $i \ne j$, $1 \le i, j \le 6$. The needed formulas are:

$$\begin{aligned} X_{23} =& (\lambda_{64} x_2 x_4 x_6 + \lambda_{45} x_3 x_4 x_5 + \lambda_{56} x_1 x_5 x_6) \frac{\partial}{\partial x_1} \\ &+ (\lambda_{16} x_1 x_2 x_6 + \lambda_{51} x_1 x_3 x_5 + \lambda_{65} x_4 x_5 x_6) \frac{\partial}{\partial x_4} \\ &+ (\lambda_{61} x_1^2 x_6 + \lambda_{14} x_1 x_3 x_4 + \lambda_{46} x_4^2 x_6) \frac{\partial}{\partial x_5} \\ &+ (\lambda_{15} x_1^2 x_5 + \lambda_{41} x_1 x_2 x_4 + \lambda_{54} x_4^2 x_5) \frac{\partial}{\partial x_6}, \end{aligned}$$

$$\begin{aligned}X_{25} =&(\lambda_{63}x_1x_3x_6 + \lambda_{34}x_3^2x_4 + \lambda_{46}x_4x_6^2)\frac{\partial}{\partial x_1}\\ &+ (\lambda_{16}x_1^2x_6 + \lambda_{41}x_1x_3x_4 + \lambda_{64}x_4^2x_6)\frac{\partial}{\partial x_3}\\ &+ (\lambda_{13}x_1x_3^2 + \lambda_{61}x_1x_6^2 + \lambda_{36}x_3x_4x_6)\frac{\partial}{\partial x_4}\\ &+ (\lambda_{31}x_1^2x_3 + \lambda_{14}x_1x_4x_6 + \lambda_{43}x_3x_4^2)\frac{\partial}{\partial x_6},\end{aligned}$$

$$\begin{aligned}X_{26} =&(\lambda_{53}x_1x_3x_5 + \lambda_{34}x_2x_3x_4 + \lambda_{45}x_4x_5x_6)\frac{\partial}{\partial x_1}\\ &+ (\lambda_{15}x_1^2x_5 + \lambda_{41}x_1x_2x_4 + \lambda_{54}x_4^2x_5)\frac{\partial}{\partial x_3}\\ &+ (\lambda_{13}x_1x_2x_3 + \lambda_{51}x_1x_5x_6 + \lambda_{35}x_3x_4x_5)\frac{\partial}{\partial x_4}\\ &+ (\lambda_{31}x_1^2x_3 + \lambda_{14}x_1x_4x_6 + \lambda_{43}x_3x_4^2)\frac{\partial}{\partial x_5},\end{aligned}$$

$$\begin{aligned}X_{35} =&(\lambda_{62}x_1x_2x_6 + \lambda_{24}x_2x_3x_4 + \lambda_{46}x_4x_5x_6)\frac{\partial}{\partial x_1}\\ &+ (\lambda_{16}x_1^2x_6 + \lambda_{41}x_1x_3x_4 + \lambda_{64}x_4^2x_6)\frac{\partial}{\partial x_2}\\ &+ (\lambda_{12}x_1x_2x_3 + \lambda_{61}x_1x_5x_6 + \lambda_{26}x_2x_4x_6)\frac{\partial}{\partial x_4}\\ &+ (\lambda_{21}x_1^2x_2 + \lambda_{14}x_1x_4x_5 + \lambda_{42}x_2x_4^2)\frac{\partial}{\partial x_6},\end{aligned}$$

$$\begin{aligned}X_{36} =&(\lambda_{52}x_1x_2x_5 + \lambda_{24}x_2^2x_4 + \lambda_{45}x_4x_5^2)\frac{\partial}{\partial x_1}\\ &+ (\lambda_{15}x_1^2x_5 + \lambda_{41}x_1x_2x_4 + \lambda_{54}x_4^2x_5)\frac{\partial}{\partial x_2}\\ &+ (\lambda_{12}x_1x_2^2 + \lambda_{51}x_1x_5^2 + \lambda_{25}x_2x_4x_5)\frac{\partial}{\partial x_4}\\ &+ (\lambda_{21}x_1^2x_2 + \lambda_{14}x_1x_4x_5 + \lambda_{42}x_2x_4^2)\frac{\partial}{\partial x_5},\end{aligned}$$

$$X_{56} =(\lambda_{23}x_1x_2x_3 + \lambda_{42}x_2x_4x_6 + \lambda_{34}x_3x_4x_5)\frac{\partial}{\partial x_1} + (\lambda_{31}x_1^2x_3 + \lambda_{14}x_1x_4x_6 + \lambda_{43}x_3x_4^2)\frac{\partial}{\partial x_2} + (\lambda_{12}x_1^2x_2 + \lambda_{41}x_1x_4x_5 + \lambda_{24}x_2x_4^2)\frac{\partial}{\partial x_3} + (\lambda_{21}x_1x_2x_6 + \lambda_{13}x_1x_3x_5 + \lambda_{32}x_2x_3x_4)\frac{\partial}{\partial x_4}.$$

It is easy to see that outside of some very special subcases of the Manakov case, all differential operators X_{ij}, $1 \le i < j \le 6$ are not identically zero. Note that $X_{ij}(H_r) = 0$, $1 \le r \le 3$, and, moreover, $X_{ij}(x_i) = X_{ij}(x_j) = 0$, $1 \le i < j \le 6$.

2.5. Linear partial differential equations. Let us consider the following linear partial differential equation

$$\sum_{i=1}^{n} a_i(x)\frac{\partial F}{\partial x_i} = 0, \tag{2.10}$$

where a_i, $1 \le i \le n$, are holomorphic functions defined on some open subset $\mathcal{U} \subset \mathbb{C}^n$.

Theorem 2.2: *Let $x_0 \in \mathcal{U}$ be such that not all $a_i(x_0)$, $1 \le i \le n$, vanish. Let us suppose that $F_1, \dots, F_{n-1}, F$ are holomorphic on $\mathcal{U}$ solutions of equation* (2.10). *Let us suppose that the vectors* $(\operatorname{grad} F_i)(x_0)$ *are linearly independent. Then there exists a neighbourhood $\mathcal{V}$ of x_0, $\mathcal{V} \subset \mathcal{U}$ and a holomorphic function Ω defined on $\mathcal{V}$, such that for every $x \in \mathcal{V}$ one has*

$$F(x) = \Omega(F_1(x), \dots, F_{n-1}(x)). \tag{2.11}$$

See §31 of [4] and also §156 of [16]. For modern treatment see the **Holomorphic Rectification Theorem** (Theorem 1.18) in [7], which immediately implies Theorem 2.2.

Further, $\mathcal{U}$ denotes a subset of $\mathbb{C}^6$ defined by the condition that for all $1 \le i < j \le 6$, and any point $z \in \mathcal{U}$, the vectors $(\operatorname{grad} H_1)(z)$, $(\operatorname{grad} H_2)(z)$, $(\operatorname{grad} H_3)(z)$, $(\operatorname{grad} x_i)(z)$, $(\operatorname{grad} x_j)(z)$ are linearly independent. Unless all λ_i, $1 \le i \le 6$, are equal, $\mathcal{U}$ is always an open dense subset of $\mathbb{C}^6$. Saying that identity (2.11) is locally fulfilled, we understand that this is so on some neighbourhood of some point from $\mathcal{U}$.

3. Proof of Theorem 1.1.

Let us suppose that the irreducible rational fraction P_1/P_2, where $P_1, P_2 \in \mathbb{C}[x_1, \ldots, x_6]$, is a first integral of system (1.2) and that P_1 (and thus also P_2) is not its first integral. Then (D1) from Section 2.1 implies that P_1 and P_2 are proper Darboux polynomials of system (1.2). Since the right-hand sides of system (1.2) are homogeneous of the same degree then from (D2) and (D4) it follows that system (1.2) admits also an irreducible homogeneous proper Darboux polynomial P and its cofactor is a homogeneous linear form, i.e.,

$$S = \sum_{i=1}^{6} \alpha_i x_1,$$

where α_i, $1 \leq i \leq 6$, are some constants. Since $S \neq 0$, then at least one of its coefficients is not zero. As explained in Section 2.2, without any loss of generality, we can assume that $\alpha_1 \neq 0$.

Theorem 1.1 is now a direct consequence of

THEOREM 3.1: *If for some $\lambda \in \mathbb{C}^6$, the Euler equations* (1.2) *have a proper Darboux polynomial then they have a polynomial fourth integral.*

Proof. The proof is quite long and it is naturally divided on three almost independent parts.

PART 1: Construction of polynomial first integral.

Let P be a proper Darboux polynomial of the Euler equations (1.2). From (2.7) and (2.8) it immediately follows that $R = PP_{(14)}$ is a Darboux polynomial of system (1.2) with cofactor $2(\alpha_1 x_1 + \alpha_4 x_4)$, i.e,

$$d(R)(x) = 2(\alpha_1 x_1 + \alpha_4 x_4)R(x). \tag{3.1}$$

Thus, from (2.9), one deduces that for the polynomial $U = R_{(25)}$

$$d(U)(x) = -2(\alpha_1 x_1 + \alpha_4 x_4)U(x),$$

and finally (see (D3) from Section 2.1) that

$$d(V) = 0,$$

where

$$V := RU = RR_{(25)} = (PP_{(14)})(PP_{(14)})_{(25)} = PP_{(14)}P_{(25)}P_{(14)(25)}.$$

This means that V is a polynomial first integral of the Euler equations (1.2).

The main difficulty is to decide when V is a fourth integral. We will prove that except for some very special subcases of the Manakov case, this is always the case. This is proved in Part 2 when the polynomials R and U are relatively prime and in Part 3 when this is not the case. As in the Manakov case, the polynomial fourth integral always exists (see Appendix), this will prove Theorem 3.1.

PART 2: R and U are relatively prime polynomials.

We have to decide when the first integrals H_1, H_2, H_3 (see (1.3)) and V are functionally independent. Let us suppose that they are functionally dependent.

Then for all $\alpha_i,\ 1 \leq i \leq 6$,

$$X_{ij}(V) = X_{ij}(R)U + X_{ij}(U)R = 0. \tag{3.2}$$

We will prove that except for very special subcases of the Manakov case, this contradicts $\alpha_1 \neq 0$.

If one supposes that polynomials R and U are relatively prime, then (3.2) shows that either R divides $X_{ij}(R)$, i.e.,

$$X_{ij}(R) = f_{ij}R, \tag{3.3}$$

where f_{ij} is a homogeneous polynomial of second degree, or $X_{ij}(R) = X_{ij}(U) = 0$. For the first possibility, according to (3.2) and (3.3), we have that

$$X_{ij}(U) = -f_{ij}U. \tag{3.4}$$

In particular, $X_{25}(R) = f_{25}R$ and $X_{25}(U) = -f_{25}U$. Applying to the first identity the change of variables τ_{25} (see Section 2.3), we conclude that $X_{25}(U) = (f_{25} \circ \tau_{25})U$ and finally that $f_{25} = -f_{25} \circ \tau_{25}$.

But this is impossible because f_{25} cannot depend on x_2 and x_5. Indeed, the maximal powers of x_2 and of x_5 in $X_{25}(R)$ respectively are never greater than their respective maximal powers in R. Thus $f_{25} = 0$ and consequently $X_{25}(R) = X_{25}(U) = 0$.

Hence we have proved that R satisfies the equation

$$X_{25}(R) = \det \frac{\partial(H_1, H_2, H_3, R)}{\partial(x_1, x_3, x_4, x_6)} = 0. \tag{3.5}$$

This is a linear homogeneous partial differential equation for R. It has five solutions H_1, H_2, H_3, x_2 and x_5 that are never functionally dependent unless

$\lambda_1 = \lambda_3 = \lambda_4 = \lambda_6$ (a subcase of the Manakov case). Thus by Theorem 2.2 (see Section 2.5), we have that locally

$$R = \Phi(H_1, H_2, H_3, x_2, x_5), \tag{3.6}$$

where Φ is some holomorphic function.

Let us note that not only $U = R \circ \tau_{25}$, but also $U = R \circ \tau_{36}$. This is so, because R is a homogeneous polynomial of even degree, and it contains only monomials that have only even sum of the powers of x_1 and x_4. Thus the monomials of R containing even sum of the powers of x_2 and x_5 contain also even sum of the powers of x_3 and x_6 and, respectively, the monomials of R containing odd sum of the powers of x_2 and x_5 contain odd sum of the powers of x_3 and x_6.

Since $U = R \circ \tau_{36}$, exactly in the same way as (3.5), one proves that $f_{36} = 0$, or equivalently that

$$X_{36}(R) = \det \frac{\partial(H_1, H_2, H_3, R)}{\partial(x_1, x_2, x_4, x_5)} = 0.$$

This equation has five solutions: H_1, H_2, H_3, x_3 and x_6 that are never functionally dependent unless $\lambda_1 = \lambda_2 = \lambda_4 = \lambda_5$ (a subcase of the Manakov case). So that locally

$$R = \Psi(H_1, H_2, H_3, x_3, x_6), \tag{3.7}$$

for some holomorphic function Ψ.

From (3.3) and (3.4) we know that

$$X_{56}(R) = f_{56} R \tag{3.8}$$

and

$$X_{56}(U) = -f_{56} U, \tag{3.9}$$

where f_{56} is a homogeneous polynomial of degree two, or $X_{56}(R) = X_{56}(U) = 0$.

We prove that f_{56} cannot depend on x_2, x_3, x_5 and x_6. Indeed, applying to identity (3.8) the change of variables τ_{25} (see (2.6)), we conclude that $X_{56}(U) = -(f_{56} \circ \tau_{25})U$. Then (3.9), leads to $f_{56} = f_{56} \circ \tau_{25}$.

Thus f_{56} either does not depend on x_2 and x_5 or is a quadratic polynomial of them. The latter is impossible because the biggest sum $\alpha + \beta$ of ${x_2}^\alpha {x_5}^\beta$ in $X_{56}(R)$ is never bigger than the same sum in R plus 1. Thus f_{56} does not depend on x_2 and x_5.

Exactly the same arguments, applied to the change of variables τ_{36}, lead to the conclusion that f_{56} does not depend on x_3 and x_6. Thus f_{56}, if it is not zero, is a homogeneous quadratic function only of x_1 and x_4.

Completely analogous considerations show that the polynomials f_{23}, f_{26} and f_{35}, if they are not zero, are homogeneous quadratic functions only of x_1 and x_4.

Assume now that at least one of the polynomials f_{56}, f_{35} and f_{26} is not zero. First, let us examine the case when $f_{56} \neq 0$. We have

$$\frac{X_{35}(R)}{X_{56}(R)} = \frac{f_{35}}{f_{56}}. \tag{3.10}$$

Hereafter, for all representations of R as a function of H_1, H_2, H_3 and two of the coordinates (see, for example, (3.6) and (3.7)) we denote by ∂_i the partial derivative with respect to i-th variable, $1 \leq i \leq 5$. We have

$$\begin{aligned} X_{35}(R) &= X_{35}(x_2)\partial_4\Phi(H_1, H_2, H_3, x_2, x_5), \\ X_{56}(R) &= X_{56}(x_2)\partial_4\Phi(H_1, H_2, H_3, x_2, x_5). \end{aligned} \tag{3.11}$$

Let us note that $\partial_4\Phi(H_1, H_2, H_3, x_2, x_5) \neq 0$ because otherwise we would have $f_{56} = 0$. Thus (3.10) leads to

$$A_1 = \frac{X_{35}(x_2)}{X_{56}(x_2)} = \frac{\lambda_{16}x_1^2x_6 + \lambda_{41}x_1x_3x_4 + \lambda_{64}x_4^2x_6}{\lambda_{31}x_1^2x_3 + \lambda_{14}x_1x_4x_6 + \lambda_{43}x_3x_4^2} = \frac{f_{35}}{f_{56}}.$$

The polynomials f_{35} and f_{56} depend only on x_1 and x_4 while A_1 depends on x_1, x_3, x_4 and x_6. Thus necessarily we have

$$\frac{\partial A_1}{\partial x_3} = 0.$$

Simple computations show that the last condition is equivalent to

$$\lambda_{13}\lambda_{16}x_1^4 - (\lambda_{14}^2 + \lambda_{16}\lambda_{43} + \lambda_{13}\lambda_{46})x_1^2x_4^2 + \lambda_{43}\lambda_{46}x_4^4 = 0. \tag{3.12}$$

Let us consider representation (3.7) of R: $R = \Psi(H_1, H_2, H_3, x_3, x_6)$ and vector field X_{26}. As above we have

$$\frac{X_{26}(R)}{X_{56}(R)} = \frac{f_{26}}{f_{56}}.$$

Taking into account that $\partial_4\Psi(H_1, H_2, H_3, x_3, x_6) \neq 0$, because $f_{56} \neq 0$, we deduce from this equation that

$$A_2 = \frac{X_{26}(x_3)}{X_{56}(x_3)} = \frac{\lambda_{15}x_1^2x_5 + \lambda_{41}x_1x_2x_4 + \lambda_{54}x_4^2x_5}{\lambda_{12}x_1^2x_2 + \lambda_{41}x_1x_4x_5 + \lambda_{24}x_2x_4^2} = \frac{f_{26}}{f_{56}}.$$

The polynomials f_{26} and f_{56} depend only on x_1 and x_4 while A_2 depends on x_1, x_2, x_4 and x_5, therefore,

$$\frac{\partial A_2}{\partial x_2} = 0.$$

The last condition is equivalent to

$$\lambda_{51}\lambda_{21}x_1^4 - (\lambda_{14}^2 + \lambda_{24}\lambda_{51} + \lambda_{21}\lambda_{54})x_1^2x_4^2 + \lambda_{54}\lambda_{24}x_4^4 = 0. \tag{3.13}$$

Let us investigate when (3.12) is fulfilled. This happens only in the following four cases:

1. $\lambda_{13} = \lambda_{43} = 0$;
2. $\lambda_{13} = \lambda_{46} = 0$;
3. $\lambda_{16} = \lambda_{43} = 0$;
4. $\lambda_{16} = \lambda_{46} = 0$.

In case 1 ($\lambda_{13} = \lambda_{43} = 0$) $X_{56}(x_2) = 0$, thus by (3.11) $X_{56}(R) = 0$ and finally $f_{56} = 0$. This contradicts our assumption that $f_{56} \neq 0$, so we do not consider this case now.

Case 2 ($\lambda_{13} = \lambda_{46} = 0$) and case 3 ($\lambda_{16} = \lambda_{43} = 0$) are particular cases of the Manakov case.

Let us consider case 4 ($\lambda_{16} = \lambda_{46} = 0$). Equating to zero, e.g., the coefficient of x_1^4 in the left hand side of (3.13) we conclude that either $\lambda_{21} = 0$ or $\lambda_{51} = 0$. Both possibilities together with the condition of case 4 lead to particular cases of the Manakov case.

When $f_{35} \neq 0$, in the same way as above we come to the following expressions:

$$B_1 = \frac{X_{56}(x_2)}{X_{35}(x_2)} = \frac{\lambda_{31}x_1^2x_3 + \lambda_{14}x_1x_4x_6 + \lambda_{43}x_3x_4^2}{\lambda_{16}x_1^2x_6 + \lambda_{41}x_1x_3x_4 + \lambda_{64}x_4^2x_6} = \frac{f_{56}}{f_{35}},$$
$$B_2 = \frac{X_{23}(x_6)}{X_{35}(x_6)} = \frac{\lambda_{15}x_1^2x_5 + \lambda_{41}x_1x_2x_4 + \lambda_{54}x_4^2x_5}{\lambda_{21}x_1^2x_2 + \lambda_{14}x_1x_4x_5 + \lambda_{42}x_2x_4^2} = \frac{f_{23}}{f_{35}}$$

therefore,

$$\frac{\partial B_1}{\partial x_3} = 0, \quad \frac{\partial B_2}{\partial x_2} = 0.$$

As in the previous case the last two equations lead to particular cases of the Manakov case.

When $f_{26} \neq 0$, we come to the expressions:

$$C_1 = \frac{X_{23}(x_5)}{X_{26}(x_5)} = \frac{\lambda_{61}x_1^2x_6 + \lambda_{14}x_1x_3x_4 + \lambda_{46}x_4^2x_6}{\lambda_{31}x_1^2x_3 + \lambda_{14}x_1x_4x_6 + \lambda_{43}x_3x_4^2} = \frac{f_{23}}{f_{26}},$$
$$C_2 = \frac{X_{56}(x_3)}{X_{26}(x_3)} = \frac{\lambda_{12}x_1^2x_2 + \lambda_{41}x_1x_4x_5 + \lambda_{24}x_2x_4^2}{\lambda_{15}x_1^2x_5 + \lambda_{41}x_1x_2x_4 + \lambda_{54}x_4^2x_5} = \frac{f_{56}}{f_{26}}$$

that give

$$\frac{\partial C_1}{\partial x_3} = 0, \quad \frac{\partial C_2}{\partial x_2} = 0.$$

These equations also lead to particular cases of the Manakov case.

Let us suppose now that $f_{26} = f_{35} = f_{56} = 0$. From the equation $X_{56}(R) = 0$ we conclude that (out of the subcase $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ of the Manakov case) locally, for some holomorphic function Θ one has

$$R = \Theta(H_1, H_2, H_3, x_5, x_6).$$

When $\partial_4\Theta(H_1, H_2, H_3, x_5, x_6) \neq 0$, the equation $X_{36}(R) = 0$ leads to

$$(\lambda_{21}x_1^2x_2 + \lambda_{14}x_1x_4x_5 + \lambda_{42}x_2x_4^2)\partial_4\Theta(H_1, H_2, H_3, x_5, x_6) = 0,$$

i.e.,

$$\lambda_{21} = \lambda_{14} = \lambda_{42} = 0. \tag{3.14}$$

On the other hand the equation $X_{26}(R) = 0$ gives

$$(\lambda_{31}x_1^2x_3 + \lambda_{14}x_1x_4x_6 + \lambda_{43}x_3x_4^2)\partial_4\Theta(H_1, H_2, H_3, x_5, x_6) = 0,$$

i.e., $\lambda_{31} = \lambda_{14} = \lambda_{43} = 0$ that, together with (3.14), leads to the already excluded case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$.

What happens when $\partial_4\Theta(H_1, H_2, H_3, x_5, x_6) = 0$? In this case we have

$$\partial_5\Theta(H_1, H_2, H_3, x_5, x_6) \neq 0,$$

because otherwise it will follow that R is functionally dependent on H_1, H_2 and H_3. But this is not so. Indeed, as follows from (3.1), R is a proper Darboux polynomial because $\alpha_1 \neq 0$. The equation $X_{25}(R) = 0$ gives

$$\left(\lambda_{31}x_1^2x_3 + \lambda_{14}x_1x_4x_6 + \lambda_{43}x_3x_4^2\right)\partial_5\Theta(H_1, H_2, H_3, x_5, x_6) = 0,$$

i.e.,

$$\lambda_{31} = \lambda_{14} = \lambda_{43} = 0. \tag{3.15}$$

The equation $X_{35}(R) = 0$ gives

$$\left(\lambda_{21}x_1^2x_2 + \lambda_{14}x_1x_4x_5 + \lambda_{42}x_2x_4^2\right)\partial_5\Theta(H_1, H_2, H_3, x_5, x_6) = 0,$$

i.e., $\lambda_{21} = \lambda_{14} = \lambda_{42} = 0$ that, together with (3.15), leads to the already excluded case. Thus the assumption that H_1, H_2, H_3 and V are functionally dependent when R and U are relatively prime can eventually be true only in some very special subcases of the Manakov case.

Remark: We have to note here that there are really some subcases of the Manakov case when our procedure does not lead to a fourth integral. For example, when $\lambda_1 = \lambda_4 = \lambda_5 = \lambda_6 = 0$ and $\lambda_2 = -\lambda_3$ (subcase of case 4) the polynomial $P = x_2 + x_3$ is a proper Darboux polynomial of the Euler equations (1.2). However, applying our procedure on P, one obtains a polynomial first integral that is functionally dependent on H_3. But we know that in the Manakov case there always exists a polynomial fourth integral (cf. Appendix). That is why we do not exclude the Manakov case from the condition of the theorem.

Part 3: R and U are not relatively prime polynomials.

We have for R and U

$$R = PP_{(14)} \quad \text{and} \quad U = P_{(25)}P_{(14)(25)}.$$

Since the polynomial P is irreducible, the polynomials $P_{(14)}$, $P_{(25)}$ and $P_{(14)(25)}$ are also irreducible.

Thus polynomials R and U are not relatively prime only in the following 8 cases:

1. $P = P_{(25)}$;
2. $P = -P_{(25)}$;
3. $P = P_{(14)(25)}$;
4. $P = -P_{(14)(25)}$;
5. $P_{(14)} = P_{(25)}$ that is equivalent to 3;
6. $P_{(14)} = -P_{(25)}$ that is equivalent to 4;
7. $P_{(14)} = P_{(14)(25)}$ that is equivalent to 1;
8. $P_{(14)} = -P_{(14)(25)}$ that is equivalent to 2.

Let us examine case 1. The cofactor of P is

$$\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4x_4 + \alpha_5x_5 + \alpha_6x_6.$$

According to (2.9) the cofactor of $P_{(25)}$ is

$$-\alpha_1 x_1 + \alpha_2 x_2 - \alpha_3 x_3 - \alpha_4 x_4 + \alpha_5 x_5 - \alpha_6 x_6.$$

P and $P_{(25)}$ are equal in the case under consideration. Comparing the two cofactors we find

$$\alpha_1 = 0, \quad \alpha_3 = 0, \quad \alpha_4 = 0, \quad \alpha_6 = 0.$$

However, this contradicts our assumption that $\alpha_1 \neq 0$. In the same way, cases 2, 3 and 4 also lead to $\alpha_1 = 0$. ∎

As an example of application of the procedure for the construction of the fourth integral described in the above proof, let us consider the product case when $\lambda_1 \neq \lambda_2$ and $\lambda_1 \neq \lambda_3$. One can easily see that in this case the polynomial

$$P = \frac{\lambda_{21}}{c} x_2 + x_3 + \frac{\lambda_{21}}{c} x_5 + x_6,$$

where $c = \sqrt{\lambda_{13}\lambda_{21}}$, is a proper Darboux polynomial of system (1.2) with cofactor $c(x_1 + x_4)$. Here, $P = P_{(14)}$ and thus $R = PP_{(14)} = P^2$ and $U = (P^2)_{(25)} = P^2_{(25)}$. Finally the polynomial

$$V = RU = \left(PP_{(25)}\right)^2 = \left[-\frac{\lambda_{21}}{\lambda_{13}}(x_2 + x_5)^2 + (x_3 + x_6)^2\right]^2$$

is a fourth integral of (1.2). In fact, in this example, $PP_{(25)}$ already is a fourth integral.

The explicit form of the polynomial fourth integral when $\lambda_2 \neq \lambda_1$ and $\lambda_2 \neq \lambda_3$ or when $\lambda_3 \neq \lambda_1$ and $\lambda_3 \neq \lambda_2$ follows now from Theorem 2.1b applied to the permutational symmetries $\tau = \tau_2 \circ \tau_3$ and τ^2, respectively.

Remark: When comparing our system (1.2) with its "twin brother"—the Euler–Poisson equations of heavy rigid body motion (see [2, 3, 14, 15, 18]) we conclude from [21] (see also [11]) that for these equations the exact counterpart of Theorem 1.1 holds. Nevertheless, the exact counterpart of Theorem 3.1 for Euler-Poisson equations fails. Indeed, in the non-integrable so-called Hess-Appelrot case, the proper Darboux polynomial exists

Appendix

Here we explicitely write down the fourth integral for the Manakov case and product case in form obtained in [9].

The table below covers all the space of parameters $(\lambda_i)_{1\leq i\leq 6}$ satisfying the Manakov condition. In this table all cases are explicitly written down, unless they can be deduced one from another by the permutational symmetry argument. The last column in this table contains necessary and sufficient conditions for functional independence of the integrals. The generic case in the table is defined explicitly by the conditions of functional independence of first integrals H_1, H_2, H_3 and F given in the last column. For the last four rows, the listed first integrals are functionally independent except for the trivial case when all components of λ are equal. The results given in this table remain valid also when $\lambda \in \mathbb{C}^6$.

Functionally independent first integrals for the Manakov case

Case		First integrals	Conditions
Generic		$H_1,\ H_2,\ H_3,$ $F = \lambda_{16}\lambda_{24}x_4^2 + \lambda_{51}\lambda_{62}x_5^2 - \lambda_{16}\lambda_{62}x_6^2$	$\lvert\lambda_{16}\rvert + \lvert\lambda_{62}\rvert > 0$ and $\lvert\lambda_{16}\rvert + \lvert\lambda_{51}\rvert > 0$ and $\lvert\lambda_{24}\rvert + \lvert\lambda_{62}\rvert > 0$ and $\lvert\lambda_{13}\rvert + \lvert\lambda_{32}\rvert > 0$
$\lambda_{16} = \lambda_{62} = 0$ (Case I)		$H_1,\ H_2,\ H_3,$ $G = x_3^2 + x_4^2 + x_5^2$	$\lvert\lambda_{43}\rvert + \lvert\lambda_{53}\rvert > 0$
	$\lambda_{43} = \lambda_{53} = 0$	$H_1,\ x_3,\ x_4,\ x_5$	no conditions
$\lambda_{16} = \lambda_{51} = 0$ (Case II)		$H_1,\ H_2,\ H_3,$ $G = \lambda_{24}\lambda_{43}x_4^2 + \lambda_{24}\lambda_{63}x_5^2 - \lambda_{43}\lambda_{62}x_6^2$	$\lvert\lambda_{43}\rvert + \lvert\lambda_{63}\rvert > 0$ and $\lvert\lambda_{43}\rvert + \lvert\lambda_{24}\rvert > 0$ and $\lvert\lambda_{24}\rvert + \lvert\lambda_{62}\rvert > 0$ and $\lvert\lambda_{13}\rvert + \lvert\lambda_{32}\rvert > 0$
	$\lambda_{43} = \lambda_{63} = 0$	$H_1,\ H_2,\ H_3,\ x_5$	no conditions
	$\lambda_{43} = \lambda_{24} = 0$	$H_1,\ H_2,\ H_3,\ x_5$	no conditions
	$\lambda_{24} = \lambda_{62} = 0$	$H_1,\ H_2,\ H_3,\ x_6$	no conditions
	$\lambda_{13} = \lambda_{32} = 0$	$H_1,\ H_2,\ H_3,\ x_1$	no conditions

In the product case, one can take as a fourth integral

$$H_4 = \lambda_1 x_1 x_4 + \lambda_2 x_2 x_5 + \lambda_3 x_3 x_6,$$

which when $(\lambda_1, \lambda_2, \lambda_3) \neq (c, c, c)$ for some $c \in \mathbb{C}$, is always functionally independent of H_1, H_2 and H_3.

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